

Non Hamiltonian Chaos from Nambu Dynamics of Surfaces¹

Minos Axenides

Institute of Nuclear Physics, NCSR Demokritos, 15310 Agia Paraskevi, Attiki, Greece
(E-mail: axenides@inp.demokritos.gr)

Abstract

We discuss recent work with E.Floratos (JHEP 1004:036,2010) on Nambu Dynamics of Intersecting Surfaces underlying Dissipative Chaos in R^3 . We present our argument for the well studied Lorenz and Rössler strange attractors. We implement a flow decomposition to their equations of motion. Their volume preserving part preserves in time a family of two intersecting surfaces, the so called *Nambu Hamiltonians*. For dynamical systems with linear dissipative sector such as the Lorenz system, they are specified in terms of Intersecting Quadratic Surfaces. For the case of the Rössler system, with nonlinear dissipative part, they are given in terms of a Helicoid intersected by a Cylinder. In each case they foliate the entire phase space and get deformed by Dissipation, the irrotational component to their flow. It is given by the gradient of a surface in R^3 specified in terms of a scalar function. All three intersecting surfaces reproduce completely the dynamics of each strange attractor.

1 Introduction

Dissipative dynamical systems, with a low dimensional phase space, present an important class of simple non-linear physical systems with intrinsic complex behavior (homoclinic bifurcations, period doubling, onset of chaos, turbulence), which generated intense experimental, theoretical and numerical work in the last few decades [1, 2].

Recently [3] we have reexamined dissipative dynamical systems with a 3-dimensional phase space from the perspective of Nambu-Hamiltonian Mechanics(NHM) [4, 5]. The latter represents a generalization of Classical Hamiltonian Mechanics, mostly appropriate for the study of odd-dimensional phase-space volume preserving flows(Liouville's theorem). As such and in order to make it directly applicable to the dynamics of dissipative systems in R^3 we must associate it to a volume preserving dynamics sector. We have done so by introducing a flow decomposition to their equations of motion and therefore isolate in their flow vector field its rotational(solenoidal) part. It is manifestly volume preserving, non-dissipative and hence directly describable in terms of the intersecting surfaces

¹Published in Chaos Theory: Modeling, Simulation and Applications,
C.H. Skiadas, I. Dimotikalis and C. Skiadas (Eds),
World Scientific Publishing Co, pp. 110-119.

of NHM. Along with the remaining irrotational component to the flow the decomposition fully recovers the dissipative dynamics of the system in question, presenting itself as an equivalent formulation of the odd-dimensional dynamical system.

We have applied the above recipe to the famous examples of Lorenz[6] and Rössler [7] chaotic attractors which represent the prototype models for the onset of turbulence.[2].

In sect. 2 we start off with a discussion of flow decomposition for the most elementary and familiar of all dissipative systems in R^2 without chaotic behavior : the dissipative harmonic oscillator (DHO). In this case the existence of a Hamiltonian formalism identifies for its "closed" nondissipative sector the harmonic oscillator (HO) with a well defined integrable classical and quantum evolution.

The presence of chaotic flows for dissipative systems in R^3 is argued, by analogy, to necessitate intersecting Nambu Surfaces which define respectively "closed" physical systems with integrable R^3 periodic orbits and simultaneously a well defined classical and quantum behaviour.

In sect. 3 we apply the framework for the cases of the Lorenz and Rössler strange attractors. We isolate their non-dissipative sector parametrized by two intersecting surfaces: a cylinder and a paraboloid for the Lorenz attractor as well as a helicoid with a cylinder for the Rössler system. They account, amazingly, for the double scroll topology of the full "butterfly" Lorenz attractor. and the single scroll topology for the Rössler case. We end our presentation with Conclusions and open problems.

2 Flow Decomposition in Dissipative Systems: R^2 versus R^3

Dissipation is a necessary condition for dynamical systems to exhibit chaos. Yet it is not sufficient. This is enunciated through a powerful No-go theorem . Indeed the Poincare-Bendixon allows for only fixed points and limits cycles in two dimensions. Chaotic Flows need space to emerge. At the minimal level they emerge with dissipative systems in R^3 . They are typically associated with Strange Attractors such as the famous ones of Lorenz and Rössler. They belong to a large class of dynamical systems whose dynamics is governed by continuous set of 1st order ordinary differential flow equations

$$\dot{\vec{x}} = \vec{v}(\vec{x}(t), t, \lambda) \quad (1)$$

where \vec{v} is a velocity field flow with λ some control external parameter. Their phase space dynamics, depending on whether they exchange energy with their environment or not, can be either *open-dissipative* or *closed (conservative-Hamiltonian)* . This is reflected on their velocity flow field being divergenceless or not. Let us see all these issues for the simple case of the Dissipative Harmonic Oscillator (DHO) whose phase space equations of motion are well known to be for $x_1 = q$ and $x_2 = p$:

$$\begin{aligned}\dot{q} &= p + \alpha q \\ \dot{p} &= -q + \beta p\end{aligned}\tag{2}$$

The velocity flow is given by $\nabla \dot{v}_{tot} = (\alpha + \beta) \neq 0$ for appropriate values $\alpha \neq -\beta \neq 0$ with a net outflow or inflow of energy depending on the sign of $\nabla \dot{v}_{tot}$ (inflow < 0 or outflow > 0). Nevertheless we split it into a nondissipative component $v_{ND} = (p, -q)$ with zero flow $\nabla \dot{v}_{ND} = 0$. Its dissipative part is given by $v_D = \alpha q + \beta p$ with its flow being simply the total flow of the HO $\nabla v_D = \alpha + \beta$. The dynamics of the HO in one dimension can take the form of a 2nd order differential equation

$$\ddot{q} + (\alpha + \beta)\dot{q} + (1 + \alpha\beta)q = 0\tag{3}$$

The damping factor is $\gamma = \alpha + \beta$ and the effective natural frequency which is dissipation induced is given by $\omega_{eff} = 1 + \alpha\beta$.

The existence of the Harmonic Oscillator Hamiltonian $H = \frac{1}{2}(p^2 + q^2)$ along with a Dissipation function $D = \frac{1}{2}(\alpha q^2 + \beta p^2)$ reproduce the eqs. of motion of (2.2) in a more compact form

$$\dot{x}^i = \epsilon^{ij}(\partial_j H + \partial_j D)\tag{4}$$

where $i, j = 1, 2$ and $\epsilon^{12} = -\epsilon^{21} = 1$ as usual. The non-dissipative sector of the HO is given by

$$\begin{aligned}\dot{q} &= \frac{\partial H}{\partial p} = p \\ \dot{p} &= -\frac{\partial H}{\partial q} = -q\end{aligned}\tag{5}$$

Firstly it identifies a circular *periodic* orbit, the famous HO as the *integrable progenitor* of the transient DHO. This is a well defined closed physical system with a most familiar quantum behavior, the Quantum Harmonic Oscillator.

Secondly the linear dissipation of the DHO a transient spirals it inwards to a fixed point of zero energy. The harmonic oscillator *localizes* in effect such a dissipative evolution.

Lastly the rate at which the energy of the harmonic oscillator loses its energy as it spirals in depends on its damping strength. It is the *Quality factor* which is easily computed to be

$$Q = \frac{\omega_{eff}}{\gamma} = \frac{1 + \alpha\beta}{\alpha + \beta}\tag{6}$$

All in all Flow decomposition for the transient DHO has three immediate implications for the DHO: 1. Existence of Integrable Harmonic oscillator associated with a periodic orbit

2. Localization of the DHO evolution and asymptotic fixed point state 3. Quality factor for DHO as a measure of its energy loss rate and in effect of the damping strenth.

We will proceed now to examine whether it is feasible to implement this methodology for the case of Strange attractors which possess chaotic flows in R^3 .

Nambu-Hamiltonian mechanics is a specific generalization of classical Hamiltonian mechanics, where the invariance group of canonical symplectic transformations of the Hamiltonian evolution equations in $2n$ dimensional phase space is extended to the more general volume preserving transformation group $SDiff(\mathcal{M})$ with an arbitrary phase space manifold M of any dimension $d = \dim(\mathcal{M})$.

In [3] we work with the case of a three dimensional flat phase space manifold. Nevertheless our results could be generalized to curved manifolds of any dimension[5, 8].

Nambu-Hamiltonian mechanics of a particular dynamical system in R^3 is defined once two scalar functions $H_i \in C^\infty(R^3), i = 1, 2$, the generalized Hamiltonians [4, 5] are provided. The evolution equations are:

$$\dot{x}^i = \{x^i, H_1, H_2\} \quad i = 1, 2, 3 \quad (7)$$

where the Nambu 3-bracket, a generalization of Poisson bracket in Hamiltonian mechanics, is defined as

$$\{f, g, h\} = \epsilon^{ijk} \partial^i f \partial^j g \partial^k h \quad i, j, k = 1, 2, 3 \quad \forall f, g, h \in C^\infty(R^3) \quad (8)$$

Any local coordinate transformation

$$x^i \rightarrow y^i = y^i(x) \quad i = 1, 2, 3 \quad (9)$$

which preserves the volume of phase space

$$\det \left(\frac{\partial y^i}{\partial x^j} \right) = 1 \quad \forall x = (x^1, x^2, x^3) \in R^3 \quad (10)$$

leaves invariant the 3-bracket and therefor it is a symmetry of Nambu-Mechanics. Except for the linearity and antisymmetry of the bracket with respect to all of its arguments it also satisfies an important identity, the so called "Fundamental identity"[FI] [3, 8].

The evolution eq.(2.7) has a flow vector field:

$$v^i(x) = \epsilon^{ijk} \partial^j H_1 \partial^k H_2 \quad i, j, k = 1, 2, 3 \quad (11)$$

which is volume preserving

$$\partial^i v^i = 0 \quad (12)$$

The reverse is also true. We will name the phase-space volume preserving flows "Non-dissipative" while the non-conserving ones "Dissipative" ($\partial^i v^i(x) > 0 (< 0)$).

The flow equations of a general dissipative system may take the general vector form

$$\dot{\vec{x}} = \vec{\nabla} H_1 \times \vec{\nabla} H_2 + \vec{\nabla} D \quad (13)$$

We notice that given a pair of functions H_1, H_2 such that

$$\vec{v}_{ND} = \vec{\nabla} H_1 \times \vec{\nabla} H_2 \quad (14)$$

any transformation of H_1, H_2

$$H_i \rightarrow H'_i(H_1, H_2) \quad i = 1, 2 \quad (15)$$

with unit Jacobian

$$\det \left(\frac{\partial H'_i}{\partial H_j} \right) = 1 \quad (16)$$

gives also

$$\vec{v}_{ND} = \vec{\nabla} H'_1 \times \vec{\nabla} H'_2 \quad (17)$$

In ref.[8] we reduced the evolution equation of the form (2.32) Nambu Mechanics in Hamiltonian-Poisson form as follows:

$$\dot{x}^i = \{x^i, H_1, \}_{H_2} \quad (18)$$

where the induced Poisson bracket

$$\{f, g\}_{H_2} = \epsilon^{ijk} \partial^i f \partial^j g \partial^k H_2 \quad (19)$$

satisfies all the required properties like linearity, antisymmetry and the Jacobi identity.

In the following sections we are going to present a detailed investigation of the Lorenz and Rössler attractors from the point of view of Dissipative Nambu-Hamiltonian Dynamics.

3 The Lorenz Attractor from Dissipative Dynamics of Intersecting Quadratic Surfaces .

The Lorenz model was invented as a three Fourier mode truncation of the basic eqs. for heat convection in fluids in Reyleigh-Benard type of experiments [12] The time evolution eqns. in the space of three Fourier modes x, y, z which we identify as phase-space are :

$$\begin{aligned} \dot{x} &= \sigma(y - x) \\ \dot{y} &= x(r - z) - y \\ \dot{z} &= xy - bz \end{aligned} \quad (20)$$

where σ is the Prandtl number, r is the relative Reynolds number and b the geometric aspect ratio .

The standard values for σ , b are $\sigma = 10, b = \frac{8}{3}$ with r taking values in $1 \leq r < \infty$. There are dramatic changes of the system as r passes through various critical values which follow the change of stability character of the three critical points of the system $P_1 : x = y = z = 0, P_{\pm} : x = y = \pm\sqrt{b(r-1)}, z = r-1$

Lorenz discovered the non-periodic deterministic chaotic orbit for the value $r = 28$, which is today identified as a Strange Attractor with a Hausdorff dimension of ($d = 2.06$)[13]. Standard reference for an exhaustive numerical investigation of the Lorenz system is the book by Sparrow [14].

There have been various attempts made to localize the Lorenz attractor, by convex surfaces, in order to get information about Hausdorff dimensions [13] and other characteristics [15].

We will proceed to exhibit Localization of the full Lorenz attractor from the Nambu surfaces which we will determine shortly. At the Quantum level the existence of an attracting ellipsoid in a matrix model formulation of the Lorenz system is manifest [3] as one gets attracting ellipsoids in higher dimensional phase spaces.

We now proceed to describe the Lorenz system in the framework of section 2. The flow vector field \vec{v} is analyzed into its dissipative and non-dissipative parts as follows:

$$\vec{v}_D = (-\sigma x, -y, -bz) = \vec{\nabla} D \quad (21)$$

with the "Dissipation" function

$$D = -\frac{1}{2} (\sigma x^2 + y^2 + bz^2) \quad (22)$$

and

$$\vec{v}_{ND} = (\sigma y, x(r-z), xy) = (0, y, z-r) \times (-x, 0, \sigma) \quad (23)$$

The two *Hamiltonians* or Clebsch-Monge potentials H_1, H_2 are determined by

$$\vec{\nabla} H_1 \times \vec{\nabla} H_2 = \vec{v}_{ND} \quad (24)$$

or equivalently

$$H_1 = \frac{1}{2} [y^2 + (z-r)^2] \quad (25)$$

and

$$H_2 = \sigma z - \frac{x^2}{2} \quad (26)$$

The Lorenz system (20) can thus be written in the equivalent form in terms of $\vec{r} = (x, y, z)$:

$$\dot{\vec{r}} = \vec{\nabla} H_1 \times \vec{\nabla} H_2 + \vec{\nabla} D \quad (27)$$

In the Non-Dissipative part(ND) of the dynamical system

$$\dot{\vec{r}} = \vec{\nabla} H_1 \times \vec{\nabla} H_2 \quad (28)$$

the Hamiltonians H_1, H_2 are conserved and their intersection defines the ND orbit. Moreover if we get the reduced Poisson structure (sect.2) from H_2 we obtain the 2-dim phase space Σ_2 to be the family of parabolic cylinders with symmetry axis the y-axis:

$$H_2 = \text{constant} = H_2(\vec{r}_o) \quad (29)$$

Σ_2 is thus given by

$$z = z_o + \frac{x^2 - x_o^2}{2\sigma} \quad (30)$$

with x_o, z_o the initial condition for x, z . The induced Poisson algebra (rel.19) is given by

$$\begin{aligned} \{x, y\}_{H_2} &= \partial_z H_2 = \sigma \\ \{y, z\}_{H_2} &= \partial_x H_2 = -x \\ \{z, x\}_{H_2} &= 0 \end{aligned} \quad (31)$$

The dynamics on the 2d-phase space Σ_2 is given by H_1

$$\begin{aligned} \dot{x} &= \{x, H_1\}_{H_2} \\ \dot{y} &= \{y, H_1\}_{H_2} \\ \dot{z} &= \{z, H_1\}_{H_2} \end{aligned} \quad (32)$$

and H_1 is an anharmonic oscillator Hamiltonian with $(x/\sigma, y)$ conjugate canonical variables. Using rel.(25-26) we get on Σ_2 :

$$H_1 = \frac{1}{2\sigma^2} [y^2 + \frac{1}{2} (x^2 - a^2)^2] \quad (33)$$

with

$$a^2 = x_o^2 - 2\sigma(z_o - r) = -2H_2 + 2\sigma r \quad (34)$$

where $\frac{1}{\sigma^2}$ plays the role of the mass. Depending on the initial conditions we may have a single well ($a^2 \leq 0, H_2 \geq \sigma r$) or a double well potential ($a^2 > 0, H_2 < \sigma r$) respectively. The trajectories, the intersections of the two cylinders, H_1 and H_2 with orthogonal symmetry axes (x, y) may either have one lobe left/right or may be running from the right to the left lobe. This is reminiscent of the topology structure of the orbits of the Lorenz chaotic attractor.

The full Lorenz system does not conserve H_1, H_2 and there is a random motion of the two surfaces against each other. Their intersection is time varying. In effect at every moment the system jumps from periodic to periodic orbit of the non-dissipative sector. Moreover the motion of the non-dissipative system around the two lobes, either left or right, can now jump from time to time from one lobe to the other.

4 The Rössler Attractor from Dissipative Dynamics of a Cylinder Intersecting with an Helicoid

Rössler introduced a simpler than Lorenz's nonlinear ODE system with a 3d- phase space, in order to study in more detail the characteristics of chaos, which is motivated by simple chemical reactions [7].

The Rössler system is given by the evolution eqns:

$$\begin{aligned}\dot{x} &= -y - z \\ \dot{y} &= x + ay \\ \dot{z} &= b + z(x - c)\end{aligned}\tag{35}$$

with parameters a, b, c usually taking standard values $a = b = 0.2, c = 5.1$ or $a = b = 0.1, c = 14$ for the appearance of the chaotic attractor.

We turn now to the study of the Rössler system as a Dissipative Nambu-Hamiltonian dynamical system.

The key difference with the Lorenz attractor is that the dynamics of the system is simpler. Chaos appears as random jumps outwards and inwards the single lob attractor.

In order to get the three scalars, the two generalized Hamiltonians H_1, H_2 which are conserved and characterize the non-dissipative part and D the dissipation term :

$$\dot{\vec{r}} = \vec{\nabla} H_1 \times \vec{\nabla} H_2 + \vec{\nabla} D\tag{36}$$

we checked after some guess work, that we must subtract and add a new term in the first equation. Indeed we find for the two parts

$$\vec{v}_{ND} = (-y - z - \frac{z^2}{2}, x, b)\tag{37}$$

$$\vec{v}_D = (\frac{z^2}{2}, ay, z(x - c))\tag{38}$$

satisfying accordingly $\vec{\nabla} \cdot \vec{v}_{ND} = 0$ and $\vec{\nabla} \times \vec{v}_D = 0$

We must determine H_1, H_2 and D such that

$$\vec{\nabla} H_1 \times \vec{\nabla} H_2 = \vec{v}_{ND}\tag{39}$$

and

$$\vec{\nabla} D = \vec{v}_D\tag{40}$$

For D we find easily

$$D = \frac{1}{2} [ay^2 + (x - c)z^2]\tag{41}$$

To get H_1, H_2 we must integrate first the Non-dissipative system:

$$\begin{aligned}\dot{x} &= -y - z - \frac{z^2}{2} \\ \dot{y} &= x \\ \dot{z} &= b\end{aligned}\tag{42}$$

The general solution is :

$$x(t) = -b(1 + z(t)) + (x_o + b(1 + z_o)) \cos(t) - (y_o + z_o + \frac{z_o^2}{2} - b^2) \sin(t)\tag{43}$$

$$y(t) = b^2 - z(t) - \frac{z^2(t)}{2} + (y_o + z_o + \frac{z_o^2}{2} - b^2) \cos(t) + (x_o + b + bz_o) \sin(t)\tag{44}$$

$$z(t) = bt + z_o\tag{45}$$

To uncover H_1, H_2 we introduce the complex variable

$$w(t) = w_1(t) + iw_2(t)\tag{46}$$

with

$$\begin{aligned}w_1(t) &= x(t) + b(1 + z(t)) \\ w_2(t) &= y(t) + z(t) + \frac{z^2(t)}{2} - b^2\end{aligned}\tag{47}$$

We obtain

$$w(t) = w_o \cdot e^{it}\tag{48}$$

with

$$w_o \equiv w(t=0)\tag{49}$$

We see that there are two constants of motion, the first one being :

$$|w(t)| = |w_o|\tag{50}$$

and we define correspondingly,

$$H_1 = \frac{1}{2} |w(t)|^2 = \frac{1}{2} (x + b(1 + z))^2 + \frac{1}{2} (y + z + \frac{z^2}{2} - b^2)^2\tag{51}$$

The second integral of motion is obtained through the phase

$$w(t) = |w_o| \cdot e^{i\varphi_o} \cdot e^{it} = |w_o| e^{i\varphi(t)}\tag{52}$$

or from (4.14)

$$\varphi(t) - \frac{z}{b} = \varphi_o - \frac{z_o}{b}\tag{53}$$

and we define appropriately the second constant surface H_2

$$H_2 = b \operatorname{arctg} \frac{y + z + \frac{z^2}{2} - b^2}{1 + b(1 + z)} - z \quad (54)$$

We easily check that rel.(41) is satisfied:

$$\vec{\nabla} H_1 \times \vec{\nabla} H_2 = (-y - z - \frac{z^2}{2}, x, b) \quad (55)$$

The family of surfaces H_1 and H_2 are a quadratic deformation of a cylinder and respectively a quadratic deformation of a right helicoid. Their intersection is the trajectory (43-44).

5 Conclusions-Open Problems

The main result of our present work is the demonstration of Dissipative Nambu-Hamiltonian mechanics of intersecting surfaces as the conceptual framework that underlies strange chaotic attractors both in their classical as well as quantum-noncommutative incarnation. It reproduces the familiar and well studied attractor dynamics of Lorenz and Rössler in a very intuitive manner accounting of their gross topological aspects (double lobe Butterfly for the Lorenz system) or single lobe for the Rössler attractor.

Quantum Nambu Dynamics of Surfaces [3, 8, 9], raises also the issue of possible existence of Quantum Strange Attractors. Their Quantum behavior was built systematically through *fuzzifying* the classical intersecting surfaces of the ND sector. We demonstrated this for the simplest case of the Lorenz system with a linear dissipation.

References

- [1] P. Cvitanovic Ed., *Universality in Chaos* Adam Holger, Bristol 1984.
- [2] J.P. Eckmann, *Roads to Turbulence in Dissipative Dynamical Systems* Rev. Mod. Phys. **53** no.4 (1981) 643.
- [3] M. Axenides and E. Floratos, *Strange Attractors in Dissipative Nambu Mechanics: Classical and Quantum Aspects* JHEP **1004** (2010) 036 [arXiv:0910.3881 [nlin.CD]].
- [4] Y. Nambu, *Generalized Hamiltonian Dynamics* Phys. Rev. D **7**, (1973) 2403.
- [5] L. Takhtajan, *On Foundation Of The Generalized Nambu Mechanics (Second Version)* Commun. Math. Phys. **160** , (1994) 295.
- [6] E.N. Lorenz, *Deterministic Non-Periodic Flow* J.Atm.Sci. **20** (1963), 130.

- [7] O.E. Rössler, *An Equation for Continuous Chaos* Phys. Lett.**57A**, (1976) 397.
- [8] M. Axenides and E. Floratos, *Nambu-Lie 3-Algebras on Fuzzy 3-Manifolds* JHEP **0902** (2009) 039 [arXiv:0809.3493 [hep-th]].
- [9] M. Axenides, E. G. Floratos and S. Nicolis, *Nambu Quantum Mechanics on Discrete 3-Tori* J. Phys. A **42** (2009) 275201 [arXiv:0901.2638 [hep-th]].
- [10] U. Weiss, *Quantum Dissipative Systems* World Scientific- Series in Modern Condensed Matter Physics, vol.**13** 2008; C-I. Um, K-H. Yeon, T.F. George, *The Quantum Damped Harmonic Oscillator* Phys. Rep. **362** (2002) 63; M.C. Gutzwiller, *Chaos in Classical and Quantum Mechanics* Springer-Verlag (1990), New York.
- [11] A. Clebsch , J.Reine Angew. Math. **56** (1859) 1 .
- [12] M. Tabor, *Chaos and Integrability in Nonlinear Dynamics: An Introduction* Wiley-Interscience (1989).
- [13] D. Farmer, E. Ott and J.A. Yorke, *Hausdorff Dimension computation* Physica **7D** (1983) 153.
- [14] C. Sparrow, *The Lorenz Equation, Bifurcations, Chaos and the Strange Attractors*, Springer-Verlag, New York 1987.
- [15] C. Doering and J. Gibbon, *On the shape and Dimension of the Lorenz Attractor* Dynamics and Stability of Systems , vol**10**, No.**3**, (1995), 255 ; *ibid*, vol**13**, No.**3**,(1998)299.